i am needed

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The need for i

Let us suppose that we are asked to find the solution of the simple equation

$$3x - 4 = 0.$$

Of course the answer is trivial: $x = \frac{4}{3}$.

We recognise the answer as being a fraction but suppose we had never heard of fractions: suppose we only knew about integers. Then we would be forced to conclude that either

- $3x - 4 = 0$ has no solution, or
- our system of integers is too small; we need a larger number system.

1. Write down some simple equations which have no solution if our number system contains only
   (a) integers,
   (b) negative integers,
   (c) rational numbers
   (d) real numbers.

The simplest answer in (d) is the equation $x^2 + 1 = 0$. No matter how hard we search, we will be unable to find a real number x which satisfies this equation. We must therefore conclude that either

- $x^2 + 1 = 0$ has no solution, or
- our real number system is too small, but a solution of $x^2 + 1 = 0$ may be found in a larger number system.

Mathematicians finally chose the second alternative. We agree to denote $\sqrt{-1}$ by the symbol i. Thus

$$i^2 = -1, \text{ or } i^2 + 1 = 0.$$

So now our equation $x^2 + 1 = 0$ has a solution, but what sort of solution is it? It is definitely not a real number, but for the moment, let us agree to treat it in exactly the same way as the real numbers, only replacing i^2 by -1 whenever it occurs.

Examples

$$i^4 = i.i.i.i = i^2.i^2 = (-1)(-1) = 1$$
$$(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$$

2. Establish the truth of each of the following expressions by expanding the left hand side.
   (a) $(1 + i)(1 - i) = 2$
   (b) $(2 + i)(3 + i) = 5 + 5i$
   (c) $i^6 = -1$
   (d) $(1 + 2i)^2 = -3 + 4i$

You probably have some doubts about all this! So did the Italian mathematician Rafael Bombelli in 1572 when he worked out the rules for these *complex numbers*. In fact, it was not until the year 1800 that a logical explanation of the meaning of i was given.

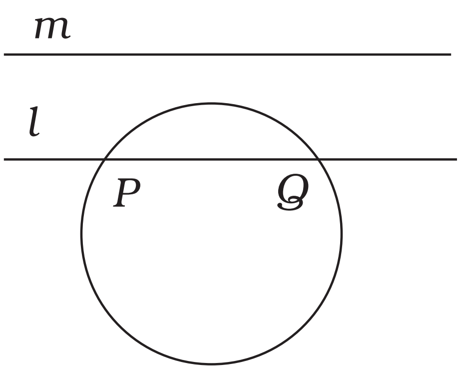
Rafael Bombelli

Some curious consequences

3. (a) By substitution, show that $2 + i$ is a solution of the (real) equation $x^2 - 4x + 5 = 0$. Is $2 - i$ a solution too?
   (b) Show that $1 + 3i$ is a solution of $x^2 - 2x + 10 = 0$. What is the other solution?
   (c) Can you give a real quadratic equation for which $1 + 2i$ is a solution?
   (d) What general behaviour seems likely here?

4. Show that the product of the complex numbers $a + bi$, $a - bi$ (where a, b are real), is a real number. What is it?

5. In the diagram below, the line l cuts the circle in the points P, Q. Does the line m cut the circle? Are you sure?



6. We find it difficult to accept i because we cannot picture what it means. It is interesting that the ancient Greeks had the same problem with negative numbers. Write a letter to Euclid telling him why you believe in negative numbers.

...but am i here?

The mysteries of mathematics are not easily revealed. Much of present day school mathematics is the product of years, sometimes centuries, of inquiring, wrestling and discovering by men of the highest intellect. The number i is no exception.


Bhaskara

In the twelfth century, the Indian mathematician Bhaskara wrote: "The square of a positive number, as also that of a negative number, is positive; and the square root of a positive number is two-fold, positive and negative; there is no square root of a negative number, for a negative number is not a square."


Wallis

In the year 1673, Wallis, while writing in defence of imaginary numbers, began: "These Imaginary Quantities (as they are commonly called) arising from the Supposed Root of a Negative Square..." His contemporary, Leibniz, believed that "imaginary numbers are a fine wonderful refuge of the Holy Spirit, a sort of amphibian between being and not being"!


Leibniz


Gauss

We notice a change of attitude in the writings of Gauss, some 60 years later.

> To integers have been added fractions, to rational quantities irrational, to positive the negative, and to the real the imaginary. This advance, however, had always been made at first with timorous and hesitating steps ... the imaginary quantities—formerly and occasionally now improperly called impossible, as opposed to real quantities—are still rather tolerated than fully naturalised ... The author has for many years considered ... [that an] objective existence can be assigned to imaginary quantities.